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Properties of Best Approximation with Interpolatory and Restricted Range Side Conditions

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An alternation property of polynomials of best uniform approximation to a function $f \in C[a, b]$ having restricted ranges of some of their derivatives is proven. For this purpose, the problem of best uniform approximation to continuous functions by polynomials having restricted ranges and satisfying interpolatory conditions on their derivatives is discussed. The method is an improved version of the one used in [3] and provides an easily computed lower bound for the number of alternations.

INTRODUCTION

In [8] and [1] the problem of best approximating a given function by polynomials with restricted ranges of some of their derivatives has been studied. Special cases of this problem are monotone approximation [6, 7] and restricted range approximation [10–12]. These papers include improved forms of Kolmogorov type theorems and theorems concerning the uniqueness of the best approximating polynomial. While in [10–12], where restrictions are imposed only on the range of the approximating polynomials, an alternation property analogous to the classical one is proven, no alternation property is given in [1, 6–8], where restrictions are imposed also on the derivatives. Even in the special case where only the range of one of the derivatives is restricted (case P_j in [6]) no alternation property is known.

The purpose of the present work is to prove an alternation property of the polynomial of best approximation (pba) in the uniform norm from the class K of polynomials having restricted ranges of their derivatives.

The proof relies on the fact that a pba from the class K is also a pba from a certain class of polynomials with restricted ranges, the derivatives of which satisfy interpolatory side conditions. Section 1 is devoted to this latter problem, which is of interest by itself, and characterization theorems, conditions for uniqueness, and an alternation property are given. These results are similar to the results in [3], where the problem of best approximation by polynomials satisfying Hermite–Birkhoff interpolation conditions is discussed.

In Section 2, the results of Section 1 are applied to the main problem of this work. A generalized alternation property of a pba from K is proven, and for the special case P_j, a lower bound for the number of alternations is given.

We assume that the reader is familiar with the concepts of Hermite–Birkhoff interpolation, its representation in terms of incidence matrices, poisedness of interpolation problems, etc., which can be found in [9].

1. Approximation by Polynomials with Restricted Ranges and Interpolatory Constraints on Their Derivatives

In this section we investigate the problem of best approximating a function $f \in C[a, b]$ in the uniform norm by polynomials in the class:

$$\hat{P} = \{p(x) \mid p(x) \in \pi_{n-1}, p^{(j)}(\xi_i) = b_{ij} \text{ when } e_{ij} = 1, l(x) \leqslant p(x) \leqslant u(x)\}, \quad (1.1)$$

where π_{n-1} is the set of all polynomials of degree $\leqslant n-1$, $E_n^k(r) = (e_{ij})_{i=1,2,\dots,k}^{j=0,1,\dots,n-1}$ is a given incidence matrix with r units describing r interpolatory conditions imposed only on the derivatives of $p(x)$ ($e_{i0} = 0$, $i = 1, 2,\dots, k$) at the points $a \leqslant \xi_1 < \xi_2 < \cdots < \xi_k \leqslant b$ (see [3]), $\{b_{ij}\}$ are fixed values, and $l(x) < u(x)$ for all $a \leqslant x \leqslant b$. The function $l(x)[u(x)]$ may take the value $-\infty$ $[\infty]$ on an open subset of $[a, b]$ and is continuous elsewhere in $[a, b]$ (see [11]). Assume that the r conditions prescribed by $E_n^k(r)$ at $\bar{\xi} = (\xi_1, \xi_2,\dots, \xi_k)$ are linearly independent on π_{n-1}. As proven in [3], such matrices satisfy a generalized Polya condition:

$$M_j \geqslant j + 1 - (n - r), \qquad j = 0, 1,\dots, n-1$$

where M_j are the Polya constants defined as:

$$M_j = \sum_{\nu=0}^{j} m_\nu \qquad j = 0, 1,\dots, n-1$$

with

$$m_\nu = \sum_{i=1}^{k} e_{i\nu} \qquad \nu = 0, 1,\dots, n-1.$$

This condition is equivalent to the following condition which does not involve r:

$$\mu_j \equiv \sum_{\nu=j}^{n-1} m_\nu = M_{n-1} - M_{j-1} \leqslant n - j, \qquad j = 0, 1,\dots, n-1, \qquad M_{-1} \equiv 0. \quad (1.2)$$

If k_1 is the index of the first nonzero column in $E_n{}^k(r)$ (in our case $k_1 \geqslant 1$) then, using (1.2) for $j = k_1$ we get:

$$r \leqslant n - k_1 . \tag{1.3}$$

The following notation for the sets containing the critical points is used in the characterization theorems, (see also [8]):

$$E_+ = E_+(p) = \{x \mid x \in [a, b], f(x) - p(x) = \| f - p \|\} \tag{1.4}$$

$$E_- = E_-(p) = \{x \mid x \in [a, b], f(x) - p(x) = - \| f - p \|\} \tag{1.5}$$

$$E_+{}^0 = E_+{}^0(p) = \{x \mid x \in [a, b], p(x) = l(x)\} \tag{1.6}$$

$$E_-{}^0 = E_-{}^0(p) = \{x \mid x \in [a, b], p(x) = u(x)\} \tag{1.7}$$

$$B = B(f, p) = E_+ \cup E_- \cup E_+{}^0 \cup E_-{}^0. \tag{1.8}$$

We assume that $f \notin \hat{P}$ and that:

$$(E_+ \cap E_-{}^0) \cup (E_- \cap E_+{}^0) = \varnothing . \tag{1.9}$$

Otherwise p is a pba to f from $\hat{P}$ and no further characterization is needed. It is easily seen that (1.9) holds if $l(x) \leqslant f(x) \leqslant u(x)$. Assume that $\hat{P}$ is not empty and, moreover, that it contains at least one polynomial that satisfies $l(x) < p(x) < u(x)$. By compactness arguments there exists a pba to f from $\hat{P}$.

The proof of the first theorem is omitted since it is similar to the proofs of the characterization theorems in [6, 7].

THEOREM 1.1. *Let $f \in C[a, b]$ and $p \in \hat{P}$ be given. Then p is a* pba *to f from $\hat{P}$ if and only if for each $p_0(x) \in \hat{P}_0$,*

$$\max_{x \in E_+ \cup E_-} [f(x) - p(x)] \, p_0(x) \geqslant 0$$

where

$$\begin{aligned} \hat{P}_0 &= \hat{P}_0(E_n{}^k(r), \xi, p) \\ &= \{p_0(x) \mid p_0(x) \in \pi_{n-1}, p_0^{(j)}(\xi_i) = 0, \qquad e_{ij} = 1, \\ &\qquad p_0(x) \geqslant 0 \text{ on } E_-{}^0 \text{ and } p_0(x) \leqslant 0 \text{ on } E_+{}^0\}. \end{aligned} \tag{1.10}$$

Another useful formulation of this theorem is:

COROLLARY 1.1. *Let $f \in C[a, b]$ and $p \in \hat{P}$ be given. Then, p is a* pba *to f from $\hat{P}$ if and only if for each $p_0 \in P_0$*

$$\max_{x \in B(f,p)} \sigma(x) \, p_0(x) \geqslant 0,$$

where

$$\begin{aligned} P_0 &= P_0(E_n{}^k(r), \bar{\xi}) \\ &= \{p_0(x) \mid p_0(x) \in \pi_{n-1}, p_0^{(j)}(\xi_i) = 0, e_{ij} = 1\} \end{aligned} \tag{1.11}$$

and $\sigma(x)$ is defined on $B(f, p)$ as:

$$\begin{aligned} \sigma(x) &= +1 \qquad x \in E_+ \cup E_+{}^0 \\ \sigma(x) &= -1 \qquad x \in E_- \cup E_-{}^0. \end{aligned} \tag{1.12}$$

For the following theorems we use some terms defined in [3]:

An L-condition is a condition corresponding to an additional unit in the first column of an incidence matrix $E_n{}^k(r)$ (possibly in a new row).

An incidence matrix $E_n{}^k(r)$ is called L-poised at a fixed point $\bar{\xi}$ with respect to the interval $[a, b]$ if for each addition of $n - r$ L-conditions in $[a, b]$, the resulting matrix describes an interpolation problem with a unique solution.

An incidence matrix $E_{\bar{n}}{}^k(\bar{r})$ is called a partial matrix of $E_n{}^k(r)$ at $\bar{\xi}$ if it contains $\bar{r}$ units corresponding to a maximal set of independent conditions on $\pi_{\bar{n}-1}$ ($\bar{n} \leqslant n$) that are prescribed by $E_n{}^k(r)$ at $\bar{\xi}$. By the last definition it follows that

$$\begin{aligned} P_0(E_{\bar{n}}{}^k(\bar{r}), \bar{\xi}) &\subset P_0(E_n{}^k(r), \bar{\xi}), \\ \hat{P}_0(E_{\bar{n}}{}^k(\bar{r}), \bar{\xi}, p) &\subset \hat{P}_0(E_n{}^k(r), \bar{\xi}, p). \end{aligned} \tag{1.13}$$

$$\bar{n} - \bar{r} \leqslant n - r. \tag{1.14}$$

It is proven in [3, Lemma 3.1], that for any ξ every incidence matrix $E_n{}^k(r)$ has at least one partial matrix that is L-poised at $\bar{\xi}$. Another result to be used later is formulated in the next lemma:

LEMMA 1.1. *Let $E_n{}^k(r)$ be an incidence matrix with r independent conditions at $\bar{\xi}$. If $E_l{}^k(M_{l-1})$, which is composed of the first l columns of $E_n{}^k(r)$, is* L-*poised at $\bar{\xi}$, and if $\mu_l = n - l$, then $E_n{}^k(r)$ is* L-*poised at $\bar{\xi}$.*

Proof. The $n - l$ conditions prescribed by $E_n{}^k(r)$ on the derivatives of order $\geqslant l$ are linearly independent on π_{n-1} and therefore the matrix $E_{n-l}^k(n - l)$ composed of the last $n - l$ columns of $E_n{}^k(r)$ is poised at $\bar{\xi}$. By adding to $E_n{}^k(r)$ any $n - r$ L-conditions we get a matrix E that can be decomposed into two matrices at the lth column, since it satisfies $\mu_l = n - l$ (or $M_{l-1} = l$) [9]. The first l columns of E is a poised matrix since it is derived from the L-poised matrix $E_l{}^k(M_{l-1})$ by addition of $n - r$ L-conditions. The last $n - l$ columns of E is the poised matrix $E_{n-l}^k(n - l)$. Thus, E is a poised matrix at $\bar{\xi}$, by [9, Lemma 4], and $E_n{}^k(r)$ is L-poised.

In addition to the above terms we introduce the following:

DEFINITION 1.1. *A matrix* $\tilde{E} = E_n^{\kappa}(\rho)$ with $e_{i0} = 0$, $i = 1,..., \kappa$ is called an LP-associate of $E_n^{k}(r)$ at $\bar{\xi}$ if there exists a point $\bar{\eta} \in E^{\kappa}$ such that $E_n^{\kappa}(\rho)$ is L-poised at $\bar{\eta}$, and

$$P_0(E_n^{\kappa}(\rho), \bar{\eta}) \subset P_0(E_n^{k}(r), \bar{\xi}). \tag{1.15}$$

From (1.15) it is obvious that

$$n - \rho \leqslant n - r, \qquad \rho \geqslant r$$

and

$$\hat{P}_0(E_n^{\kappa}(\rho), \bar{\eta}, p) \subset \hat{P}_0(E_n^{k}(r), \bar{\xi}, p). \tag{1.16}$$

The following lemma shows that every matrix has an LP-associate at $\bar{\xi}$:

LEMMA 1.2. *Each matrix $E_n^{k}(r)$ has an* LP-*associate matrix at $\bar{\xi}$ with $\rho = n - k_1$, where k_1 is the index of the first nonzero column in $E_n^{k}(r)$.*

Proof. Let $E_n^{\kappa}(\rho)$ be a matrix composed of first k_1 zero columns, that constitute an L-poised matrix, and $n - k_1$ columns that constitute any poised matrix at a point $\bar{\eta} \in R^{\kappa}$. Each polynomial $p_0(x) \in P_0(E_n^{\kappa}(\rho), \bar{\eta})$ is of degree $\leqslant k_1 - 1$, since by the structure of the last $n - k_1$ columns of $E_n^{k}(\rho)$ $p_0^{(k_1)}(x) \equiv 0$. Therefore $p_0(x)$ satisfies automatically all the homogeneous conditions prescribed by $E_n^{k}(r)$ at $\bar{\xi}$ that are imposed only on the derivatives of order $\geqslant k_1$, and

$$P_0(E_n^{\kappa}(\rho), \bar{\eta}) \subset P_0(E_n^{k}(r), \bar{\xi}).$$

By Lemma 1.1, $E_n^{\kappa}(\rho)$ is L-poised at $\bar{\eta}$, and therefore, it is an LP-associate of $E_n^{k}(r)$ at $\bar{\xi}$.

DEFINITION 1.2. A matrix $E_n^{\kappa}(\rho)$ is called a best LP-associate of $E_n^{k}(r)$ at $\bar{\xi}$, if among all LP-associates of $E_n^{k}(r)$ at $\bar{\xi}$ it has the minimal number of units.

By Definition 1.1 and the fact that $\rho \geqslant r$, it follows that an L-poised matrix at $\bar{\xi}$ is a best LP-associate of itself at $\bar{\xi}$.

THEOREM 1.2. *Let P^* be the set of all* pba *to $f \in C[a, b]$ from $\hat{P}$ and let $E_n^{\kappa}(\rho)$ be a best* LP-*associate of $E_n^{k}(r)$ at $\bar{\xi}$. Then the set $B^* = \bigcap_{p \in P^*} B(f, p)$ contains at least $n + 1 - \rho$ points, and all $p \in P^*$ coincide on this set.*

Proof. First we prove that for each $p \in P^*$, $B(f, p)$ contains $n + 1 - \rho$ points. Suppose that for some $p \in P^*$, $B(f, p)$ contains less than $n + 1 - \rho$ points. (Note that by Lemma 1.2 $\rho \leqslant n - k_1$ so that $n - \rho \geqslant k_1 \geqslant 1$.)

It is possible to construct a polynomial $p_0(x) \in \pi_{n-1}$ that satisfies:

$$p_0^{(j)}(\eta_i) = 0 \qquad e_{ij} = 1 \text{ in } E_n{}^{\kappa}(\rho) \tag{1.17}$$

$$p_0(x) \geqslant 0 \qquad \text{on } E_-{}^0 \tag{1.18}$$

$$p_0(x) \leqslant 0 \qquad \text{on } E_+{}^0 \tag{1.19}$$

$$p_0(x) = -(f(x) - p(x)) \qquad \text{on} \quad E_+ \cup E_- . \tag{1.20}$$

This follows from the L-poisedness of $E_n{}^{\kappa}(\rho)$ at $\bar{\eta}$, and from the fact that conditions (1.17)–(1.20) are the ρ conditions prescribed by $E_n{}^{K}(\rho)$ at $\bar{\eta}$, and at most $n - \rho$ additional L-conditions at points of $B(f, p)$. Note that in view of (1.9) there is no contradiction among conditions (1.18)–(1.20); moreover, since $e_{i0} = 0$, $i = 1,..., k$ in $E_n{}^{\kappa}(\rho)$, there is no overlapping between condition (1.17) and the rest.

By (1.17)–(1.19), $p_0(x) \in \hat{P}_0(E_n{}^{\kappa}(\rho), \bar{\eta}, p)$ and by (1.16), $p_0(x) \in \hat{P}_0(E_n{}^k(r), \bar{\xi}, p)$ as well. But it follows from (1.20) that

$$\max_{x \in E_+ \cup E_-} (f(x) - p(x))\, p_0(x) < 0$$

which is a contradiction of Theorem 1.1. Thus, each set $B(f, p)$ where $p \in P^*$, contains at least $n + 1 - \rho$ points. By the convexity of the set P^*, there is a polynomial $p^*(x) \in P^*$ such that for every $p_1(x) \in P^*$ there is a $p_2(x) \in P^*$ and two scalars α, β satisfying

$$p^*(x) = \alpha p_1(x) + \beta p_2(x) \qquad \alpha, \beta > 0 \quad \alpha + \beta = 1 \tag{1.21}$$

(see [13, p. 16, Theorem 4]). From (1.21), the triangle inequality, and the fact that each $p(x) \in P^*$ satisfies $l(x) \leqslant p(x) \leqslant u(x)$ it follows easily that for each $p \in P^*$

$$E_+(p^*) \subset E_+(p),\ E_-(p^*) \subset E_-(p),\ E_+{}^0(p^*) \subset E_+{}^0(p),\ E_-{}^0(p^*) \subset E_-{}^0(p) \tag{1.22}$$

and therefore $B^* = B(f, p^*)$. Hence B^* contains at least $n + 1 - \rho$ points, and for any $p \in P^*$

$$p(x) = p^*(x) \qquad x \in B(f, p^*) = B^*. \tag{1.23}$$

This completes the proof of the theorem.

It is obvious that Theorem 1.2 holds for any LP-associate of $E_n{}^k(r)$, but by taking a best LP-associate the guaranteed number of points in B^* is maximal. In view of this remark and Lemma 1.2 we get a lower bound for the number of points in $B^*(f, p)$:

COROLLARY 1.2. *The set $B^*(f, p)$ contains at least $k_1 + 1$ points.*

The lower bound for the number of points in $B^*(f, p)$ given by Corollary 1.2 is easily determined from the matrix $E_n^k(r)$. An improved lower bound, which also is easily found, is derived below from a known result of Atkinson and Sharma [9 Tl m 6], which states that an incidence matrix with $r = n$ (a full matrix) satisfies the Polya condition ($M_j \geqslant j + 1$, $j = 0,\ldots, n - 1$) and contains no supported odd blocks (A-S condition), is poised.

LEMMA 1.3. *Let $E_n^k(r_1)$ be the matrix derived from $E_n^k(r)$ by addition of one unit at the end of each odd block that does not terminate at the last column and is not prescribed at the points $\{a, b\}$, and let $\tilde{M}_i$, $i = 0,\ldots, n - 1$ be the Polya constants defined with respect to $E_n^k(r_1)$. Then there exists an* LP-*associate of $E_n^k(r)$ at $\bar{\xi}$ with respect to $[a, b]$ that contains ρ units*:

$$\rho = \tilde{M}_{l-1} + n - l \leqslant n - k_1$$

where

$$l = \min\{j \mid j - \tilde{M}_{j-1} \geqslant i - \tilde{M}_{i-1}\, , i = 1,\ldots, n\}.$$

Proof. First we show that the first l columns of $E_n^k(r_1)$ constitute a matrix $E_l^k(\tilde{M}_{l-1})$ that is L-poised at $\bar{\xi}$. The last column of $E_l^k(\tilde{M}_{l-1})$ is a zero column. Otherwise $\tilde{M}_{l-1} \geqslant \tilde{M}_{l-2} + 1$, in contradiction to the definition of l. Thus, all the blocks in $E_l^k(\tilde{M}_{l-1})$ prescribed at points of (a, b) are of even length. $E_l^k(\tilde{M}_{l-1})$ satisfies the generalized Polya condition (1.2) with, strict inequality since for all $j < l$, $l - \tilde{M}_{l-1} > j - \tilde{M}_{j-1}$, and hence $\tilde{M}_{l-1} - \tilde{M}_{j-1} < l - j$. Therefore, the addition of $l - \tilde{M}_{l-1}$ L-conditions to $E_l^k(\tilde{M}_{l-1})$ in $[a, b]$ results in a poised matrix (it satisfies the Polya condition and the A-S condition) which shows that $E_l^k(\tilde{M}_{l-1})$ is L-poised. The matrix $E_n^\kappa(\rho)$ composed of $E_l^k(\tilde{M}_{l-1})$ as the first l columns and any $n - l$ columns that constitute a full poised matrix is an LP-associate of $E_n^k(r)$ at $\bar{\xi}$ by its structure and by Lemma 1.1. In this matrix

$$\rho = \tilde{M}_{l-1} + n - l,$$

and since by the definition of l: $l - \tilde{M}_{l-1} \geqslant k_1 - \tilde{M}_{k_1-1} = k_1$ it follows that $\rho \leqslant n - k_1$.

Theorem 1.2 is analogous to [3, Theorem 3.2] in the sense that the role played by $B(f, p)$ here is the same as that of $A(f, p)$ there. But here we use a best LP-associate of $E_n^k(r)$ at $\bar{\xi}$, $E_n^\kappa(\rho)$, while in [3] we used the maximal partial matrix of $E_n^k(r)$, which is L-poised at $\bar{\xi}$, denoted by $E_{\bar{n}}^k(\bar{r})$. The present result is stronger since the matrix $E_n^k(r_0)$, the first $\bar{n}$ columns of which are those of $E_{\bar{n}}^k(\bar{r})$ and its last $n - \bar{n}$ columns constitute a full poised matrix at $\bar{\xi}$, is by Lemma 1.1 an LP-associate of $E_n{}^k(r)$ at $\bar{\xi}$. By the structure of $E_n{}^k(r_0)$:

$$P_0(E_{\bar{n}}{}^k(\bar{r}), \bar{\xi}) = P_0(E_n{}^k(r_0), \bar{\xi}) \tag{1.24}$$

and

$$\bar{n} - \bar{r} = n - r_0 \leqslant n - \rho. \tag{1.25}$$

In view of the above discussion some of the theorems in [3] can be improved, but the examples that appear there to show the sharpness of the theorems can still serve since for them $\bar{n} - \bar{r} = n - \rho$.

The following example shows that a case in which $\bar{n} - \bar{r} < n - \rho$ can occur:

EXAMPLE 1.1. Let $E_n{}^k(r) = E_4{}^1(1) = (0100)$ be an incidence matrix defined at $\bar{\xi} = (0)$. It is easily seen that this matrix is not L-poised at $\bar{\xi}$ with respect to any interval containing zero as an interior point. The maximal partial matrix of $E_4{}^1(1)$ which is L-poised at $\bar{\xi}$ is (01) for which $\bar{n} - \bar{r} = 2 - 1 = 1$. But following the construction of Lemma 1.3, we see that the matrix $(0110) = E_4{}^1(2)$ is an LP-associate of $E_4{}^1(1)$ at $\bar{\xi}$, for which $n - \rho = 4 - 2 = 2 > 1$.

Using Theorem 1.2 in the same manner as Theorem 3.2 of [3] is used there, results concerning relations between polynomials in P^* are easily proven. These relations depend on the structure of $E_n{}^k(r)$ or its partial matrices; the proofs will be omitted.

THEOREM 1.3. *Let $\bar{n}$ be the maximal integer for which $E_{\bar{n}}{}^k(\bar{r})$ is an* L*-poised partial matrix of $E_n{}^k(r)$ at $\bar{\xi}$. Then for any two distinct polynomials $p_1(x)$, $p_2(x) \in P^*$, the difference $p_1(x) - p_2(x)$ is of degree $\geqslant \bar{n}$.*

COROLLARY 1.3. *Among all* pba *to $f(x)$ from $\hat{P}$ there exists at most one of degree $\leqslant \bar{n} - 1$.*

COROLLARY 1.5. *If $E_n{}^k(r)$ is* L*-poised at $\bar{\xi}$ then there is a unique* pba *to f from $\hat{P}$.*

In [3] there are two more theorems (Theorems 3.4, 3.5) that give sufficient conditions for uniqueness and are also valid in our case, if the set $A(f, p)$ there is replaced by the set $B(f, p)$ here. We omit the formulations and the proofs of these theorems.

The concluding part of this section deals with the alternation property of a pba from the class $\hat{P}$. First we prove:

LEMMA 1.4. *If $E_n{}^k(r)$ is* L*-poised at $\bar{\xi}$ with respect to $[a, b]$ and satisfies*

$e_{i0} = 0$, $i = 1, 2,..., k$, *then the space* $P_0 = P_0(E_n^k(r), \bar{\xi})$ *is a Haar space of dimension* $n - r$ *over* $[a, b]$.

Proof. Since the r conditions prescribed by $E_n^k(r)$ at $\bar{\xi}$ are linearly independent, P_0 is of dimension $n - r$. Suppose there is a nonzero polynomial $p \in P_0$ with at least $n - r$ zeroes in $[a, b]$. This polynomial satisfies the r conditions prescribed by $E_n^k(r)$ at $\bar{\xi}$, and $n - r$ L-conditions in $[a, b]$, in contradiction to the assumption that $E_n^k(r)$ is L-poised.

This lemma is a special case of [3, Lemma 4.2].

THEOREM 1.4. *Let* $f \in C[a, b]$, $l(x) \leqslant f(x) \leqslant u(x)$, *and* $p \in P^*$, *and let* $E_n^\kappa(\rho)$ *be a best* LP-*associate of* $E_n^k(r)$ *at* $\bar{\xi}$. *Then there are at least* $n + 1 - \rho$ *points of* $B(f, p)$, $a \leqslant x_1 < \cdots < x_{n+1-\rho} \leqslant b$ *where* $\sigma(x)$, *defined in* (1.12), *satisfies* $\sigma(x_{i+1}) = -\sigma(x_i)$, $i = 1, 2,..., n - \rho$.

Proof. Suppose there is a $p \in P^*$ with only l alternations of the above type, with $1 \leqslant l \leqslant n - \rho$, at the points $a \leqslant x_1 < x_2 < \cdots < x_l \leqslant b$. For every $1 \leqslant i \leqslant l - 1$ we define:

$$y_i' = \sup\{y \mid y \in [x_i, x_{i+1}] \cap B(f, p), \sigma(y) = \sigma(x_i)\},$$

$$y_i'' = \inf\{y \mid y \in [x_i, x_{i+1}] \cap B(f, p), \sigma(y) = \sigma(x_{i+1})\}.$$

By the continuity of the functions involved it follows that $\sigma(y_i') = \sigma(x_i)$ and $\sigma(y_i'') = \sigma(x_{i+1}) = -\sigma(x_i)$. Moreover, $y_i' < y_i''$ since if $y_i' > y_i''$ there are more than l alternations while $y_i' \neq y_i''$ follows from the fact that $l(x) < u(x)$ and $l(x) \leqslant f(x) \leqslant u(x)$. Let us choose in every interval (y_i', y_i'') a point y_i, $i = 1, 2,..., l - 1$. By Lemma 1.4, $P_0(E_n^\kappa(\rho), \bar{\eta})$ is a Haar space over $[a, b]$, and by [2, Theorem 5.2, p. 30] it is possible to construct a polynomial $p_0 \in P_0(E_n^\kappa(\rho), \bar{\eta})$ which has exactly $l - 1$ simple zeroes at the points $\{y_i\}$, $i = 1,..., l - 1$, and is nonzero elsewhere in $[a, b]$. By taking either $+p_0(x)$ or $-p_0(x)$ we get that $p_0(x)\, \sigma(x) < 0$ on $B(f, p)$, and in view of (1.15) this contradicts Corollary 1.1.

In case $E_n^k(r)$ is L-poised at $\bar{\xi}$, then by the last theorem there are at least $n + 1 - r$ points in $B(f, p)$. The sufficiency of the alternation property is proven only for L-poised matrices.

THEOREM 1.5. *Let* $E_n^k(r)$ *be* L-*poised at* $\bar{\xi}, f \in C[a, b]$, $l(x) \leqslant f(x) \leqslant u(x)$, *and let* $p^* \in \hat{P}$ *be a polynomial such that* $B(f, p^*)$ *contains* $n + 1 - r$ *consecutive points* $a \leqslant x_1 < x_2 < \cdots < x_{n+1-r} \leqslant b$ *where* $\sigma(x_{i+1}) = -\sigma(x_i)$, $i = 1,..., n - r$. *Then* p^* *is the* pba *to* f *from* $\hat{P}$.

Proof. Let p_0 be the pba to $f - p^*$ from the class:

$$Q = \{q \mid q \in \pi_{n-1}, q \in P_0(E_n^k(r), \bar{\xi}), l - p^* \leqslant q \leqslant u - p^* \text{ in } [a, b]\}$$

Then $p^* + p_0$ is the pba to f from $\hat{P}$. (The pba is unique by Corollary 1.5).

By Lemma 1.4, $P_0(E_n{}^k(r), \bar{\xi})$ is a Haar space over $[a, b]$ and by the alternation property for the case of restricted ranges [11], $p_0 = 0$ is the pba to $f - p^*$ from Q. Therefore p^* is the pba to f from $\hat{P}$.

2. Alternation Property of pba with Restricted Ranges of Derivatives

In this section we use results of the previous section to prove an alternation property of the pba to $f \in C[a, b]$ from the class:

$$K = \{p \mid p \in \pi_{n-1}, l_i(x) \leqslant p^{(k_i)}(x) \leqslant u_i(x), x \in [a, b], i = 0, 1,..., s\}$$

where

$$0 \leqslant k_0 < k_1 < \cdots < k_s \leqslant n - 1.$$

A Kolmogorov type characterization theorem for such a pba is proven in [8, Theorem 2] under the following assumptions:

(a) $l_i(x) < u_i(x)$, $x \in [a, b]$, $i = 0,..., s$.

(b) $l_i(x)[u_i(x)]$ may take the value $-\infty[+\infty]$ on an open subset $X_i^-[X_i^+]$ of $[a, b]$.

(c) On $[a, b] - X_i^-$ $[[a, b] - X_i^+]$ $l_i(x)[u_i(x)]$ is continuous.

(d) K is nonempty and there exists a polynomial $p \in K$ satisfying $l_i(x) < p^{(k_i)}(x) < u_i(x)$, $x \in [a, b]$, $i = 0,..., s$.

To formulate this characterization theorem further notation is needed:

$$\begin{aligned} E_+{}^i(p) &= \{x \mid x \in [a, b], p^{(k_i)}(x) = l_i(x)\}, \qquad i = 0,..., s \\ E_-{}^i(p) &= \{x \mid x \in [a, b], p^{(k_i)}(x) = u_i(x)\}, \qquad i = 0,..., s. \\ K_0(p) &= \{p_0 \mid p_0 \in \pi_{n-1}, p_0^{(k_i)}(x) \geqslant 0, x \in E_-{}^i(p), \\ &\qquad p_0^{(k_i)}(x) \leqslant 0, x \in E_+{}^i(p), i = 0,..., s\}. \end{aligned} \tag{2.1}$$

THEOREM 2.1. *Let $f \in C[a, b]$ and $p \in K$. Then p is a* pba *to f from K if and only if for each $p_0 \in K_0(p)$*

$$\max_{x \in E_+(p) \cup E_-(p)} [f(x) - p(x)]\, p_0(x) \geqslant 0. \tag{2.2}$$

[$E_+(p)$ and $E_-(p)$ are defined in (1.4) *and* (1.5)].

A direct consequence of this theorem is formulated in:

THEOREM 2.2. *Let* $p(x)$ *be a* pba *to* $f(x)$ *from* K *such that the sets of points* $E_+^{i_j}$, $j = 1,\ldots, \nu$ *and* $E_-^{l_j}$, $j = 1,\ldots, \mu$, $0 \leqslant \nu, \mu \leqslant s$, *are empty. Then* $p(x)$ *is a* pba *to* $f(x)$ *from the class*

$$\Omega = \{p \mid p \in \pi_{n-1}, l_i(x) \leqslant p^{(k_i)}(x), x \in [a, b], i = 0,\ldots, s$$
$$i \neq i_1,\ldots, i_\nu, p^{(k_i)} \leqslant u_i(x), x \in [a, b], i = 0,\ldots, s, i \neq l_1,\ldots, l_\mu\}.$$

Proof. By Theorem 2.1 each $p_0 \in K_0(p)$ satisfies (2.2), and by assumption $K_0(p) = \Omega_0(p)$, where

$$\Omega_0(p) = \{p_0 \mid p_0 \in \pi_{n-1}, p_0^{(k_i)}(x) \geqslant 0, x \in E_-{}^i(p),$$
$$i = 0,\ldots, s, i \neq l_1,\ldots, l_\mu, p_0^{(k_i)} \leqslant 0, x \in E_+{}^i(p),$$
$$i = 0,\ldots, s, i \neq i_1,\ldots, i_\nu\}.$$

Using Theorem 2.1 in the opposite direction we get that $p(x)$ is also a pba to $f(x)$ from Ω.

Without loss of generality we assume that $k_0 = 0$ (taking $-l_0(x) = u_0(x) = M$, M large enough, in case there are no restrictions on the range of the values of the approximating polynomials).

With each $p \in K$ we associate the following class of polynomials:

$$\hat{P}(p) = \{q \mid q \in \pi_{n-1}, q^{(k_i)}(x) = l_i(x), x \in E_+{}^i(p),$$
$$q^{(k_i)}(x) = u_i(x), x \in E_-{}^i(p), i = 1,\ldots, s, l_0(x) \leqslant q(x) \leqslant u_0(x)\}. \quad (2.3)$$

From the set of conditions

$$q^{(k_i)}(x) = \begin{cases} l_i(x), & x \in E_+{}^i(p) \\ u_i(x), & x \in E_-{}^i(p) \end{cases} \qquad i = 1,\ldots, s$$

we take a maximal set of r conditions that are independent on π_{n-1} $(r \leqslant n - k_1 \leqslant n - 1)$;

$$q^{(j)}(\xi_i) = b_{ij}, \qquad e_{ij} = 1, \qquad i = 1,\ldots, k, \quad j = 1,\ldots, n-1 \quad (2.4)$$

Here $e_{ij} = 1$ only if $j = k_\nu$ for some $1 \leqslant \nu \leqslant s$, $\xi_i \in E_+{}^\nu \cup E_-{}^\nu$ and b_{ij} is either $l_\nu(\xi_i)$ or $u_\nu(\xi_i)$. These conditions can be described by an incidence matrix $E_n{}^k(r) = (e_{ij})$ at the point $\bar{\xi} = (\xi_1,\ldots, \xi_k)$ with $e_{i0} = 0$, $i = 1,\ldots, k$. In terms of this incidence matrix, $\hat{P}(p)$ in (2.3) is given by

$$\hat{P}(p) = \{q \mid q \in \pi_{n-1}, q^{(j)}(\xi_i) = b_{ij}, e_{ij} = 1, l_0(x) \leqslant q(x) \leqslant u_0(x)\}, \quad (2.5)$$

and is of the same type as $\hat{P}$ of the previous section.

THEOREM 2.3. *Let $p(x)$ be a* pba *to $f(x)$ from K, then $p(x)$ is also a* pba *to $f(x)$ from $\hat{P}(p)$.*

Proof. By Theorem 2.1, for each $p_0(x) \in K_0(p)$, (2.2) holds. Let $E_n{}^k(r)$ and $\bar{\xi}$ be defined as in (2.4) with respect to $p(x)$, and let $\hat{P}_0(p)$ be the class

$$\hat{P}_0(p) = \{p_0 \mid p_0 \in \pi_{n-1}, p_0^{(j)}(\xi_i) = 0, e_{ij} = 1, p_0(x) \geqslant 0 \\ x \in E_-{}^0(p), p_0(x) \leqslant 0, x \in E_+{}^0(p)\}. \tag{2.6}$$

By the choice of $E_n{}^k(r)$ and $\bar{\xi}$

$$\hat{P}_0(p) \subset K_0(p), \tag{2.7}$$

and thus, each $p_0 \in \hat{P}_0(p)$ satisfies (2.2). In view of Theorem 1.1 this condition is sufficient for $p(x)$ to be a pba to $f(x)$ from $\hat{P}(p)$.

This theorem enables us to apply Theorem 1.4 and get the following alternation property of a pba from K:

THEOREM 2.4. *Let $p(x)$ be a* pba *to $f(x)$ from K, where $l_0(x) \leqslant f(x) \leqslant u_0(x)$, and let $E_n{}^k(r)$ be an incidence matrix that defines $\hat{P}(p)$ as in* (2.5). *If $E_n{}^\kappa(\rho)$ is a best* LP-*associate of $E_n{}^k(r)$, at $\bar{\xi}$, then there are $t = n + 1 - \rho$ points $a \leqslant x_1 < \cdots < x_t \leqslant b$ in $B(f, p)$ [defined in* (1.8)] *such that $\sigma(x)$ [defined in* (1.12)] *satisfies*:

$$\sigma(x_{i+1}) = -\sigma(x_i), \qquad i = 1, 2,..., t - 1.$$

For the case $s = 1$ with $k_1 = j$ (case P_j in [6] when $-l_0(x) = +u_0(x) = M$, M large enough) all columns of $E_n{}^k(r)$ are zero except the jth column in which there are r units, where by (1.2) $0 \leqslant r \leqslant n - j$. In the next theorem we give a lower bound for the number t in Theorem 2.4, which is easily determined in this special case:

THEOREM 2.5. *Let $E_n{}^k(r)$ be the incidence matrix corresponding to a* pba *from K in case $s = 1$, $k_1 = j$, and let θ, $0 \leqslant \theta \leqslant 2$, be the number of units in $E_n{}^k(r)$ corresponding to conditions at end points $\{a, b\}$. Then the number t in Theorem* 2.4 *is bounded below by* $\max\{n + 1 - 2r + \theta, j + 1\}$.

Proof. Let $E_n{}^k(r_1)$ and $\tilde{M}_i$, $i = 0,..., n - 1$, be defined as in Lemma 1.3. Then

$$\begin{aligned} \tilde{M}_i &= 0, && i = 0,..., j - 1 \\ \tilde{M}_i &= r, && i = j \\ \tilde{M}_i &= 2r - \theta, && i = j + 1,..., n - 1 \end{aligned}$$

and

$$l = \min\{j \mid j - \tilde{M}_{j-1} \geqslant i - \tilde{M}_{i-1}, i = 1,..., n\}$$

is either j or n. Therefore by Lemma 1.3

$$t = n + 1 - \rho \geqslant \max\{n + 1 - 2r + \theta, j + 1\}.$$

The following example demonstrates the sharpness of the two last theorems.

EXAMPLE 2.1. Let $K = \{p \mid p \in \pi_2, p'(x) \leqslant 1, -1 \leqslant x \leqslant 1\}$. The pba to $f(x) = x^2$ from K is $p(x) = \frac{1}{2}x^2 + \frac{1}{4}$, as can be verified by a direct application of the sufficiency part of Theorem 2.1. Here $p'(x) = x$, $E_-^1 = \{1\}$, $E_+(p) \cup E_-(p) = \{-1, 0, 1\}$, and

$$K_0(p) = \{p_0 \mid p_0 \in \pi_2, p'(1) \geqslant 0\}.$$

Therefore, there is no polynomial $p_0 \in K_0$ that satisfies $p_0(-1) < 0$, $p_0(0) > 0$, $p_0(1) < 0$. The incidence matrix for this problem is $E_3^1(1) = (0\ 1\ 0)$ with the condition $p'(1) = u_1(1) = 1$ prescribed at the end point of the interval.

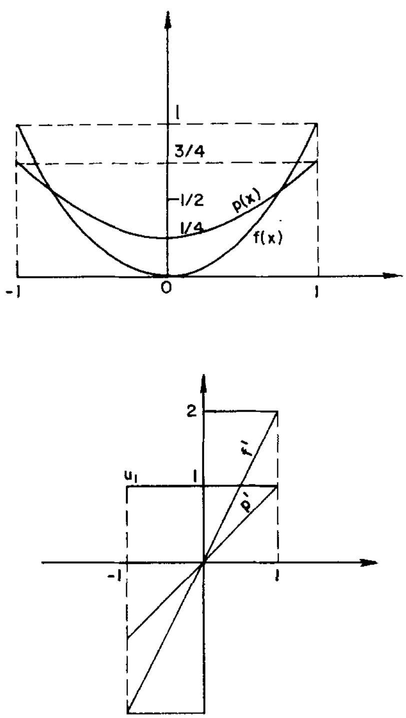

FIGURE 1.

Therefore it is L-poised at $\bar{\xi} = (1)$ with $\rho = r = 1$, $\theta = 1$, and number t of Theorem 2.4 is $n + 1 - \rho = 3$, which is the exact number of alternations of $f - p$ in this case (see Fig. 1). In this example the lower bound given for the number of alternations by Theorem 2.5 is also achieved, since $n + 1 - 2r + \theta = 3$.

With the method of this section we get only the necessity of the alternation property of a pba to $f(x)$ from K. Obviously a sufficient condition that is not necessary for $p(x)$ to be a pba to $f(x)$ from K is that the set $B(f, p)$ contains $n + 1$ points $a \leqslant x_1 < \cdots < x_{n+1} \leqslant b$ such that $\sigma(x_{i+1}) = -\sigma(x_i)$, $i = 1,..., n$. In this case, $p(x)$ is also a pba to $f(x)$ from the class

$$\Phi = \{p(x) \mid p \in \pi_{n-1}, l_0(x) \leqslant p(x) \leqslant u_0(x), x \in [a, b]\}. \tag{2.8}$$

To generalize Remes type algorithms for the construction of pba from the class K, the gap between the necessity and sufficiency of the alternation property must be closed.

The results of [3] and the method of this paper can be combined to study the problem of best approximation from the class K, when equalities between $l_i(x)$ and $u_i(x)$, $i = 0,..., s$ are permitted, as is done in [12] for the case $s = 0$. We intend to investigate this problem in the future.

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